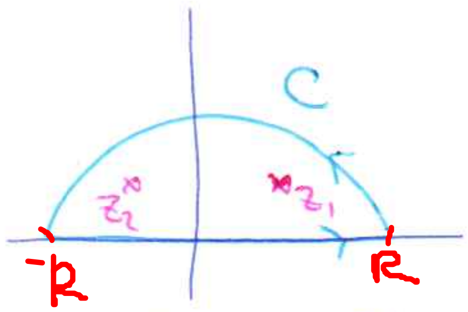
Cálculo de integrales impropias.

(A) $$\int_{-\infty}^{\infty} \underbrace{\frac{1}{1+x^4}}_{\text{acotada en } [a,b] \text{, para todo } a,b} dx$$

Por comparación con $\frac{1}{x^4}$, la integral converge.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx = VP\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^4}\,dx$$

Sea $f(z) = \frac{1}{1+z^4}$, y C: semicirc $|z|=R$ y, $\operatorname{Im} z \geq 0$,
y segmento $[-R,R]$, eje real



$$\int_C f(z)\,dz = 2\pi i\left(\sum \operatorname{Res}(f, z_j)\right)$$

Sing: $z^4 = -1$

$$e^{4i\theta} = e^{-i\pi} \Rightarrow \theta = \frac{-\pi + 2k\pi}{4}$$

$z_0 = e^{-\frac{\pi}{4}i}$, $z_1 = e^{\frac{\pi}{4}i}$, $z_2 = e^{\frac{3\pi}{4}i}$, $z_3 = e^{\frac{5\pi}{4}i}$

$$\underbrace{\int_C f(z)\,dz}\; = 2\pi i\left(\operatorname{Res}(f, z_1) + \operatorname{Res}(f, z_2)\right) \underset{\text{clase 11/6}}{=} \frac{\pi\sqrt{2}}{2}$$

$$\int_{\Gamma_R} f(z)\,dz + \int_{-R}^{R} \underbrace{\frac{1}{1+x^4}\,dx}_{\substack{z=x \\ dz=dx}} = \frac{\pi\sqrt{2}}{2} \quad *$$

Γ_R semicirc.

$$\left|\int_{\Gamma_R} \frac{1}{1+z^4}\,dz\right| \leq \frac{1}{R^4-1}\cdot \pi R \xrightarrow{R\to\infty} 0$$

Tomando límite en $*$: $VP\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^4}\,dx = \pi\frac{\sqrt{2}}{2}$

Similarmente se aplica el procedimiento p/ integrar funciones racionales $\frac{P(x)}{Q(x)}$ con $gr(P) \leq gr(Q) - 2$ y Q no tiene raíces reales.

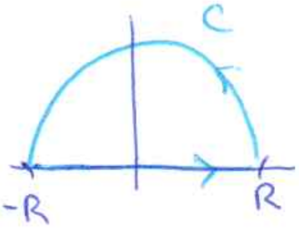
(B) $\displaystyle\int_0^\infty \underbrace{\frac{\cos(ax)}{(x^2+b^2)}}_{\text{acotada en } \mathbb{R}} dx \qquad a,b>0$

Converge absolutamente, ya que $0 \le \left|\frac{\cos(ax)}{x^2+b^2}\right| \le \frac{1}{x^2+b^2} < \frac{1}{x^2}$

y $\int_1^\infty \frac{1}{x^2}dx$ (C) $\Rightarrow \int_1^\infty \left|\frac{\cos(ax)}{x^2+b^2}\right|dx$ (C) $\Rightarrow \int_1^\infty \frac{\cos(ax)}{x^2+b^2}$ (C.A.) $\Rightarrow$

$\int_0^\infty \frac{\cos(ax)}{x^2+b^2}dx$ (C).

¿Cómo calculamos? Si hacemos $f(z) = \frac{\cos(az)}{z^2+b^2}$ y calculamos:



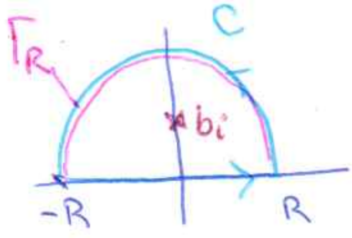
$$\int_C f(z) = \underbrace{\int_{\Gamma_R} \frac{\cos(az)}{z^2+b^2}dz}_{\substack{\text{no se puede acotar,} \\ \text{porque } \cos(az) \text{ no está} \\ \text{acotado en semiplano} \\ \text{superior (ni en el inferior)}}} + \int_{-R}^{R} \frac{\cos(ax)}{x^2+b^2}dx$$

Recordemos: $|\cos(az)| = |\cos(ax)\cdot ch(ay) - i\,sen(ax)\,sh(ay)|$

si $z = yi$: $|\cos(az)| = ch(ay) \to$ no acotado

Mejor: $\tilde{f}(z) = \frac{e^{iaz}}{z^2+b^2} = \frac{\cos(az)}{z^2+b^2} + i\frac{sen(az)}{z^2+b^2}$

si $z = x$: $\tilde{f}(x) = \underbrace{\frac{\cos(ax)}{x^2+b^2}}_{\substack{\text{función de la integral} \\ \text{pedida.}}} + i\frac{sen(ax)}{x^2+b^2}$



$(R > b)$

Calculamos:

$$\int_C \tilde{f}(z) = \int_C \frac{e^{iaz}}{z^2+b^2}\,dz = 2\pi i \,\mathrm{Res}(\tilde{f}, bi)$$

bi: polo simple de $\tilde{f}$

$$\mathrm{Res}(\tilde{f}, bi) = \lim_{z\to bi} \frac{(z-bi)\,e^{iaz}}{(z-bi)(z+bi)} = \frac{e^{-ab}}{2bi}$$

(✱) $$\int_C \frac{e^{iaz}}{z^2+b^2}\,dz = 2\pi i\,\frac{e^{-ab}}{2bi} = \underbrace{\int_{\Gamma_R} \frac{e^{iaz}}{z^2+b^2}\,dz}_{\to 0 \text{ cdo } R\to\infty} + \underbrace{\int_{-R}^{R} \frac{e^{iax}}{x^2+b^2}\,dx}_{\substack{z=x\\ dz=dx}}$$

$$\left|\int_{\Gamma_R} \frac{e^{iaz}}{z^2+b^2}\,dz\right| \leq \sup\left\{\left|\frac{e^{iaz}}{z^2+b^2}\right|, z\in\Gamma_R\right\}\cdot \underbrace{\pi R}_{\text{long. } \Gamma_R} \leq \frac{1}{R^2-b^2}\cdot\pi R \xrightarrow{R\to\infty} 0$$

$$\left|\frac{e^{iaz}}{z^2+b^2}\right| \leq \frac{|e^{i(ax+iay)}|}{\big||z|^2-b^2\big|} \underset{z\in\Gamma_R}{=} \frac{e^{-ay}}{R^2-b^2} \leq \frac{1}{R^2-b^2}$$

$y \geq 0$ en Γ_R, $e^{-ay} \leq 1$

Luego: de (✱), tomando $R\to\infty$:

$$\frac{\pi e^{-ab}}{b} = \mathrm{VP}\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+b^2}\,dx = \mathrm{VP}\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+b^2}\,dx + i\,\mathrm{VP}\int_{-\infty}^{\infty} \frac{\operatorname{sen} ax}{x^2+b^2}\,dx$$

$$\Rightarrow \mathrm{VP}\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx = \frac{\pi e^{-ab}}{b} \Rightarrow \int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx = 2\int_0^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx = \frac{\pi e^{-ab}}{b}$$

$$\mathrm{VP}\int_{-\infty}^{\infty} \frac{\operatorname{sen}(ax)}{x^2+b^2}\,dx = 0$$

Similar procedimiento se aplica a integrales de la forma:

$$\int \cos(ax) f(x)\,dx \ \text{ o } \int \operatorname{sen}(ax) f(x) \quad \text{donde} \quad R\,|f(z)| \xrightarrow[R\to\infty]{} 0, \ z\in\Gamma_R$$

$\frac{P}{Q}$ gr $Q \geq$ gr $P + 2$

c) $\displaystyle\int_{-\infty}^{\infty} \underbrace{\frac{x}{x^2+b^2}\,\text{sen}(ax)}_{\text{acotada.}}\, dx \qquad a>0$

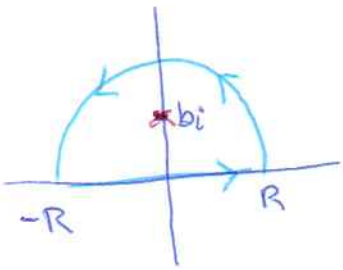
Converge: porque el integrando es $\displaystyle\underbrace{\frac{x}{x^2+b^2}}_{\substack{\text{monótona}\\ \text{en } [1,\infty)\\ \lim_{x\to\infty}\frac{x}{x^2+b^2}=0}} \cdot \underbrace{\text{sen}(ax)}_{\substack{\text{tiene integrales}\\ \text{acotadas:}\\ \left|\int_1^{\beta}\text{sen}(ax)\right| \le \frac{2}{a}}}$

$\Rightarrow$ por Dirichlet-Abel, $\displaystyle\int_1^{\infty}\frac{x}{x^2+b^2}\,\text{sen}(ax)\,dx$ converge.

$\displaystyle\int_{-\infty}^{-1}\frac{x}{x^2+b^2}\,\text{sen}(ax)\,dx$ converge, porque el integrando es par

$\displaystyle\int_{-1}^{1}\frac{x}{x^2+b^2}\,\text{sen}(ax)\,dx$ converge porque integrando es continuo.

Calculamos: si usamos $\displaystyle\tilde{f}(z)=\frac{e^{iaz}}{z^2+b^2}\cdot z$



$$\int_C \tilde{f}(z)\,dz = \int_C \frac{e^{iaz}\cdot z}{z^2+b^2}\,dz = 2\pi i\,\text{Res}(\tilde{f}, bi)$$

$$\text{Res}(\tilde{f}, bi) = \lim_{z\to bi} \frac{e^{iaz}\cdot z\,(z-bi)}{(z-bi)(z+bi)} = \frac{e^{-ab}\cdot bi}{2bi} = \frac{e^{-ab}}{2}$$

(*) $$\Rightarrow \int_C \tilde{f}(z)\,dz = \underbrace{\int_{\Gamma_R}\frac{e^{iaz}\cdot z}{z^2+b^2}\,dz}_{\xrightarrow[R\to\infty]{} 0\ ?} + \int_{-R}^{R}\frac{e^{iax}\cdot x}{x^2+b^2}\,dx = 2\pi i\,\frac{e^{-ab}}{2}$$

$$\left|\int_{\Gamma_R}\frac{e^{iaz}z}{z^2+b^2}\,dz\right| \le \sup\left\{\left|\frac{e^{iaz}z}{z^2+b^2}\right|,\ z\in\Gamma_R\right\}\pi R \le \frac{R}{R^2-b^2}\cdot\pi R \longrightarrow \pi$$

ups!...

$$\left|\frac{e^{iaz}\cdot z}{z^2+b^2}\right| \le \frac{e^{-ay}\cdot|z|}{\left||z|^2-b^2\right|} \underset{z\in\Gamma_R,\ |z|=R}{=} e^{-ay}\frac{R}{R^2-b^2} \underset{y\ge 0 \text{ en } \Gamma_R \Rightarrow e^{-ay}\le 1}{\le} \frac{R}{R^2-b^2}$$

Acotemos mejor:

Lema de Jordan:

Sea $\varphi(z)$ tal que $\sup\{|\varphi(z)|,\ z\in\Gamma_R\} \xrightarrow{R\to\infty} 0$, φ holo en semiplano superior, excepto un nº finito de singularidades

siendo Γ_R: semicircunferencia en semiplano superior.

y sea $a>0$.

Entonces $\displaystyle\int_{\Gamma_R} \varphi(z)\cdot e^{iaz}\,dz \xrightarrow{R\to\infty} 0$

En nuestro caso: $\varphi(z)=\dfrac{z}{z^2+b^2}$ satisface: $|\varphi(z)|=\left|\dfrac{z}{z^2+b^2}\right|\leq \dfrac{R}{R^2-b^2} \xrightarrow{R\to\infty} 0$

$$\Rightarrow \int_{\Gamma_R} e^{iaz}\frac{z}{z^2+b^2}\,dz \xrightarrow{R\to\infty} 0$$

Queda, tomando límite $R\to\infty$ en (*):

$$VP\int_{-\infty}^{\infty} e^{iax}\frac{x}{x^2+b^2}\,dx = VP\int_{-\infty}^{\infty}\frac{\cos(ax)\,x}{x^2+b^2}\,dx + VP\int_{-\infty}^{\infty} i\,\frac{\operatorname{sen}(ax)}{x^2+b^2}\,dx = \pi e^{-ab}\cdot i$$

$$\Rightarrow VP\int_{-\infty}^{\infty}\frac{\cos(ax)}{x^2+b^2}\,x\,dx = 0$$

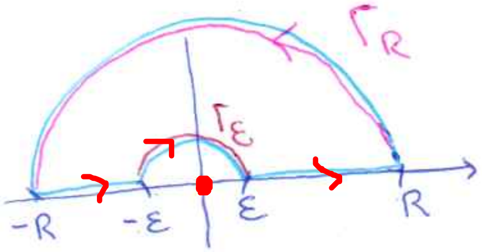
$$VP\int_{-\infty}^{\infty}\frac{\operatorname{sen}(ax)}{x^2+b^2}\,dx = \int_{-\infty}^{\infty}\frac{\operatorname{sen}(ax)\cdot x}{x^2+b^2}\,dx = \pi e^{-ab}$$

Se aplica similarmente p/ $\int_{-\infty}^{\infty}\cos(ax)\cdot f(x)\,dx$ o $\int_{-\infty}^{\infty}\operatorname{sen}(ax)\,f(x)\,dx$ con $f=\frac{P}{Q}$, racional, $gr(Q)>gr(P)$

D) $\displaystyle\int_0^{\infty}\frac{\operatorname{sen} ax}{x}\,dx \qquad a>0$

Converge! (Por criterio de Dirichlet)

Sea $\tilde{f}(z)=\dfrac{e^{iaz}}{z}$



$\tilde{f}$ tiene polo simple en $z=0$. ~~Debe~~

$$\int_C \tilde{f}(z)\,dz = \int_C \frac{e^{iaz}}{z}\,dz = 0$$ porque es holo en $C \cup RI(C)$.

$$0 = \int_C \frac{e^{iaz}}{z}\,dz = \int_{\Gamma_R} \frac{e^{iaz}}{z}\,dz + \int_{-R}^{-\varepsilon} \frac{e^{iax}}{x}\,dx + \int_{\Gamma_\varepsilon} \frac{e^{iaz}}{z}\,dz + \int_{\varepsilon}^{R} \frac{e^{iaz}}{z}\,dz$$

$\Rightarrow$:

$\varphi(z) = \frac{1}{z}$ es tal que $|\varphi(z)| = \frac{1}{|z|} \underset{z\in\Gamma_R}{=} \frac{1}{R} \xrightarrow{R\to\infty} 0$

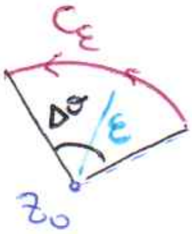
$$\Rightarrow \int_{\Gamma_R} \frac{e^{iaz}}{z}\,dz \xrightarrow{R\to\infty} 0$$

$\int_{\Gamma_\varepsilon} \frac{e^{iaz}}{z}\,dz$? Usamos el resultado:

<u>Teorema</u>: Si z_0 es polo simple de f

$$\lim_{\varepsilon\to 0} \int_{C_\varepsilon} f(z)\,dz = \text{Res}(f,z_0)\cdot i\,\Delta\theta$$

siendo C_ε: arco de circ. centrado en z_0, radio ε, de longitud $\Delta\theta\cdot\varepsilon$ recorrido sentido antihorario.



<u>Dem</u>: $f(z) = \frac{\overbrace{c_{-1}}^{\text{Res}(f,z_0)}}{z-z_0} + c_0 + c_1(z-z_0) + \cdots$ DSL.

C_ε: $z = z_0 + \varepsilon e^{it}$ $t\in[\theta_0,\theta_1]$

$$\int_{C_\varepsilon} f(z)\,dz = \int_{C_\varepsilon} \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + \ldots\, dz =$$

$$= \int_{\theta_0}^{\theta_1} \left(\frac{c_{-1}}{\varepsilon e^{it}} + c_0 + c_1 \varepsilon e^{it} + c_2 \varepsilon^2 e^{i2t} + \ldots \right) \cdot \varepsilon i e^{it}\,dt =$$

$$= i\int_{\theta_0}^{\theta_1} \left(c_{-1} + \varepsilon e^{it} c_0 + \varepsilon^2 e^{2it} c_1 + \varepsilon^3 e^{2it} c_2 + \ldots \right) dt =$$

$$= i\int_{\theta_0}^{\theta_1} c_{-1}\,dt + i\varepsilon \int_{\theta_0}^{\theta_1} e^{it} c_0 + \varepsilon e^{2it} c_1 + \varepsilon^2 e^{2it} c_2 + \ldots\, dt \xrightarrow{\varepsilon\to 0} i c_{-1}(\theta_1 - \theta_0)$$

$i c_{-1} \Delta\theta$

En nuestro ejemplo:

× recorrido durante

$$\int_{\Gamma_\varepsilon} \frac{e^{iaz}}{z}\,dz \xrightarrow{\varepsilon\to 0} \underbrace{\mathrm{Res}\left(\frac{e^{iaz}}{z},0\right)}_{e^{ia0}=1} \cdot i \cdot \pi \cdot (-1) = -\pi i \cdot 1$$

Luego, de: $0 = \int_{\Gamma_R} + \int_{-R}^{-\varepsilon} + \int_{\Gamma_\varepsilon} + \int_{\varepsilon}^{R}$,

tomando límite cuando $R\to\infty$, $\varepsilon\to 0$:

$$0 = VP\int_{-\infty}^{\infty} \frac{e^{iax}}{x}\,dx - i\pi$$

$$\Rightarrow VP\int_{-\infty}^{\infty} \frac{e^{iax}}{x}\,dx = VP\int_{-\infty}^{\infty} \frac{\cos(ax)}{x}\,dx + i\,VP\int_{-\infty}^{\infty} \frac{\operatorname{sen}(ax)}{x}\,dx = i\pi$$

$$\Rightarrow VP\int_{-\infty}^{\infty} \frac{\cos(ax)}{x}\,dx = 0 \qquad \int_1^{\infty} \frac{\cos ax}{x}\ (C) \qquad \int_0^1 \frac{\cos x}{x}\ \text{N.C.}$$

$$VP\int_{-\infty}^{\infty} \frac{\operatorname{sen}(ax)}{x}\,dx = \int_{-\infty}^{\infty} \frac{\operatorname{sen}(ax)}{x}\,dx = \pi \Rightarrow \int_0^{\infty} \frac{\operatorname{sen}(ax)}{x}\,dx = \frac{\pi}{2}$$

Ojo: eso si $a>0$ (si $a\le 0$ no valen los límites usados con $R\to\infty$)

Si $a<0$: sea $A=-a>0$

$$\int_0^{\infty} \frac{\operatorname{sen}(ax)}{x}\,dx = \int_0^{\infty} \frac{\operatorname{sen}(-Ax)}{x}\,dx = -\int_0^{\infty} \frac{\operatorname{sen}(Ax)}{x}\,dx \underset{A>0}{=} -\frac{\pi}{2}$$

(E) $\displaystyle\int_0^{\infty} \frac{\ln x}{x^2+1}\,dx$

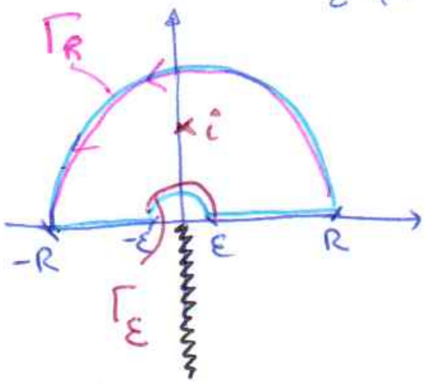
Convergencia:

si $x\approx 0$: $\left|\frac{\ln x}{x^2+1}\right| \le |\ln x|$ y $\int_0^1 |\ln x|\,dx = -\int_0^1 \ln x\,dx$ converge (clase 19)

$\Rightarrow \int_0^1 \frac{\ln x}{x^2+1}\,dx$ converge.

y $\int_1^{\infty} \frac{\ln x}{x^2+1}\,dx$ converge (clase 19)

Sea $\tilde{f}(z) = \dfrac{\log z}{z^2+1}$ con $\log z = \ln|z| + i \arg z$ $\quad -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$



(*) $\displaystyle\int_C \frac{\log z}{z^2+1}\,dz = 2\pi i \operatorname{Res}(\tilde{f}, i) = 2\pi i \frac{\log(i)}{2i} = \pi \cdot i\frac{\pi}{2}$

$$\int_C = \int_{\Gamma_R} + \int_{-R}^{-\varepsilon} + \int_{\Gamma_\varepsilon} + \int_{\varepsilon}^{R}$$

Sobre Γ_R: $\left|\tilde{f}(z)\right| = \left|\dfrac{\log(z)}{z^2+1}\right| \leq \dfrac{|\ln|z| + i\arg z|}{\big||z|^2-1\big|} \leq \dfrac{\ln R + \pi}{R^2-1}$

- $\displaystyle\left|\int_{\Gamma_R} \tilde{f}(z)\,dz\right| \leq \frac{(\ln R + \pi)}{R^2-1}\cdot \pi R \xrightarrow{R\to\infty} 0$

Sobre Γ_ε: $|\tilde{f}(z)| \leq \dfrac{|\ln \varepsilon| + \pi}{|\varepsilon^2-1|}$

- $\displaystyle\left|\int_{\Gamma_\varepsilon} \tilde{f}(z)\,dz\right| \leq \frac{(|\ln\varepsilon| + \pi)}{|\varepsilon^2-1|}\cdot \pi\varepsilon \xrightarrow{\varepsilon\to 0} 0$

- $\displaystyle\int_{-R}^{-\varepsilon} \frac{\log z}{z^2+1}\,dz \underset{\downarrow}{=} \int_R^{\varepsilon} \frac{\ln|-x| + i\arg(-x)}{(-x)^2+1}(-dx) = \int_\varepsilon^R \frac{\ln x + i\pi}{x^2+1}\,dx =$

$z = -x$, $dz = -dx$

$$= \int_\varepsilon^R \frac{\ln x}{x^2+1}\,dx + i\pi\int_\varepsilon^R \frac{1}{x^2+1}\,dx$$

$$\int_0^\infty \frac{1}{x^2+1}\,dx = \frac{\pi}{2}$$

Volviendo a (*)

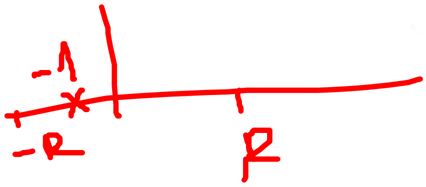
$$\int_{\Gamma_R} \frac{\log z}{z^2+1}\,dz + \int_{\Gamma_\varepsilon} \frac{\log z}{z^2+1}\,dz + 2\int_\varepsilon^R \frac{\ln x}{x^2+1}\,dx + i\pi\int_\varepsilon^R \frac{1}{x^2+1}\,dx = \pi^2\frac{i}{2}$$

Con $R\to\infty$, $\varepsilon\to 0$ resulta:

$$2\int_0^\infty \frac{\ln x}{x^2+1}\,dx + i\pi\cdot\frac{\pi}{2} = \pi^2\frac{i}{2}$$

$$\Rightarrow \int_0^\infty \frac{\ln x}{x^2+1}\,dx = 0$$

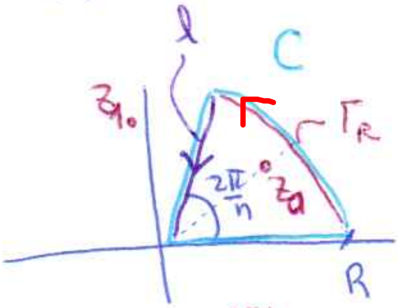
(F) $\displaystyle\int_0^\infty \frac{1}{x^n+1}\,dx$, n impar, $n>1$



Ojo: el integrando NO es impar:

$$f(x)=\frac{1}{x^n+1} \Rightarrow f(-x)=\frac{1}{(-x)^n+1}=\frac{1}{-x^n+1}\neq -f(x)$$

Converge la integral, por comparación con $\frac{1}{x^n}$ $(n>1)$
(f es ~~acotada~~ continua en $[0,\infty)$)



$z_0 = e^{i\pi/n}$

$$\int_C f(z)\,dz=\int_C \frac{1}{z^n+1}\,dz = 2\pi i \operatorname{Res}(f, z_0)$$

Sing: $z^n=-1$

$$e^{in\theta}=e^{i\pi} \Rightarrow \theta=\frac{\pi}{n}+\frac{2k\pi}{n}$$

$k=0$: $z_0=e^{i\pi/n}$

$k=1$: $z_1=e^{i\frac{3\pi}{n}}$

$\Big\}$ polos simples

$$\operatorname{Res}(f, z_0)=\lim_{z\to z_0}(z-z_0)\frac{1}{z^n+1}=\lim_{z\to z_0}\frac{1}{nz^{n-1}}=\frac{1}{nz_0^{n-1}}$$

$$\int_C=\int_0^R \frac{1}{x^n+1}\,dx+\int_{\Gamma_R}\frac{1}{z^n+1}\,dz+\int_\ell \frac{1}{z^n+1}\,dz=\frac{2\pi i}{nz_0^{n-1}} \quad (*)$$

$$\left|\int_{\Gamma_R}\frac{1}{z^n+1}\,dz\right| \leq \frac{1}{R^n-1}\cdot R\cdot\frac{2\pi}{n}\xrightarrow{R\to\infty}0$$

En ℓ: $z=t\cdot e^{i\frac{2\pi}{n}}$ $\quad dz=e^{i\frac{2\pi}{n}}dt$ $\quad$, $t\in[0,R]$

$$\int_\ell \frac{1}{z^n+1}\,dz=-\int_0^R \frac{1}{t^n e^{i2\pi}+1}\cdot e^{i\frac{2\pi}{n}}dt=-e^{i\frac{2\pi}{n}}\int_0^R\frac{1}{t^n+1}\,dt$$

$(*)$: $$\int_0^R \frac{1}{x^n+1}\,dx+\int_{\Gamma_R}\frac{1}{z^n+1}\,dz-e^{i2\pi/n}\int_0^R\frac{1}{x^n+1}\,dx=\frac{2\pi i}{nz_0^{n-1}}$$

Con $R\to\infty$:
$$\left(1-e^{i\frac{2\pi}{n}}\right)\int_0^\infty \frac{1}{x^n+1}\,dx=\frac{2\pi i}{nz_0^{n-1}}$$

$$\int_0^\infty \frac{1}{x^n+1}\,dx = \frac{1}{\left(1-e^{i\frac{2\pi}{n}}\right)} \cdot \frac{2\pi i}{n \cdot e^{i\frac{\pi}{n}(n-1)}} = \frac{2\pi i}{\left(1-e^{i\frac{2\pi}{n}}\right) \cdot n \underbrace{e^{i\pi}}_{-1} \cdot e^{-\frac{i\pi}{n}}}$$

$$= \frac{2\pi i}{\left(e^{i\frac{\pi}{n}} - e^{-i\frac{\pi}{n}}\right) \cdot n} = \frac{2\pi i}{n \cdot 2i\,\text{sen}\left(\frac{\pi}{n}\right)} = \frac{\pi/n}{\text{sen}(\pi/n)}$$

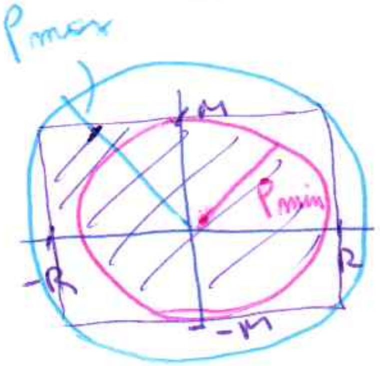
⑥ $\int_{-\infty}^{\infty} e^{-x^2}\,dx$.

Convergencia → se deja de tarea al lector.

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \lim_{R\to\infty} \int_{-R}^{R} e^{-x^2}\,dx$$

$$I^2 = \lim_{R\to\infty} \int_{-R}^{R} e^{-x^2}\,dx \cdot \lim_{M\to\infty} \int_{-M}^{M} e^{-y^2}\,dy =$$

$$= \lim_{R\to\infty} \lim_{M\to\infty} \int_{-R}^{R}\int_{-M}^{M} e^{-x^2} e^{-y^2}\,dy\,dx = \lim_{\substack{R\to\infty \\ M\to\infty}} \underbrace{\int_{-R}^{R}\int_{-M}^{M} e^{-(x^2+y^2)}\,dy\,dx}_{\text{a polares}\ldots}$$



$$\int_0^{2\pi}\int_0^{\rho_{min}} e^{-\rho^2}\rho\,d\rho\,d\theta \le \int_{-R}^{R}\int_{-M}^{M} e^{-(x^2+y^2)}\,dy\,dx \le \int_0^{2\pi}\int_0^{\rho_{max}} e^{-\rho^2}\rho\,d\rho\,d\theta$$

$$2\pi\left(\frac{e^{-\rho_{min}^2}}{-2} + \frac{1}{2}\right) \le \ldots\ldots \le 2\pi\left(\frac{e^{-\rho_{max}^2}}{-2} + \frac{1}{2}\right)$$

Cuando $R\to\infty$ y $M\to\infty$, $\rho_{min}\to\infty$; $\rho_{max}\to\infty$

Resulta:

$$\frac{2\pi}{2} \le \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy}_{I^2} \le \frac{2\pi}{2}$$

$$\Rightarrow I^2 = \pi \quad \Rightarrow I = \sqrt{\pi}.$$

(G') $\int_{-\infty}^{\infty} e^{-ax^2}\,dx$ con $a>0$

$$t=\sqrt{a}\,x \qquad dt=\sqrt{a}\,dx \quad \Rightarrow \int_{-\infty}^{\infty} e^{-ax^2}dx = \int_{-\infty}^{\infty} e^{-t^2}\cdot\frac{dt}{\sqrt{a}} = \frac{\sqrt{\pi}}{\sqrt{a}}$$

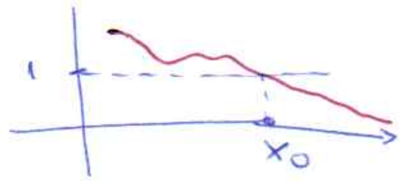
$$\boxed{\int_{-\infty}^{\infty} e^{-ax^2}dx = \frac{\sqrt{\pi}}{\sqrt{a}}}$$

(H) $\int_0^{\infty} e^{-x}x^p\,dx = \Gamma(p+1) \to$ función Gamma.

Converge? para qué p?

- Como $e^{-x}x^p = \dfrac{e^{-x}\cdot x^{p+2}}{x^2}$ y $\lim\limits_{x\to+\infty} e^{-x}x^{p+2} = 0$,

existe x_0 tal que $e^{-x}x^{p+2}<1$ para $x>x_0$



Para $x>x_0$: $e^{-x}\cdot x^p < \dfrac{1}{x^2}$

Como $\int_{x_0}^{\infty}\dfrac{1}{x^2}dx$ converge $\Rightarrow \int_{x_0}^{\infty} e^{-x}\cdot x^p dx$ converge.

- si $x\approx 0$, $e^{-x}x^p \sim x^p$

$$\lim_{x\to 0^+}\frac{e^{-x}\cdot x^p}{x^p} = 1 \Rightarrow \int_0^{x_0} e^{-x}x^p dx\ (c) \Leftrightarrow \underbrace{\int_0^{x_0} x^p dx\ (c)}_{\text{esto ocurre para } p>-1}$$

$$\Rightarrow \int_0^{\infty} e^{-x}x^p dx\ (c) \Leftrightarrow p>-1.$$

$\Gamma(\mu) = \int_0^{\infty} e^{-x}x^{\mu-1}dx \to$ bien definida para $\mu>0$

$$\Gamma(1) = \int_0^{\infty} e^{-x}dx = \frac{e^{-x}}{-1}\Big|_0^{\infty} = 1$$

$$\Gamma(\mu+1) = \int_0^{\infty} e^{-x}x^{\mu}dx \underset{\substack{\downarrow \\ \text{por partes}}}{=} -x^{\mu}\cdot e^{-x}\Big|_0^{\infty} + \int_0^{\infty} e^{-x}\cdot\mu\, x^{\mu-1}dx = \mu\,\Gamma(\mu)$$

$\Gamma(2) = \Gamma(1+1) = 1.\Gamma(1) = 1$

$\Gamma(3) = \Gamma(2+1) = 2.\Gamma(2) = 2.1$

$\Gamma(4) = \Gamma(3+1) = 3\,\Gamma(3) = 3.2.1$

$\vdots$

$\Gamma(n) = (n-1)!$ si $n \in \mathbb{N}$.

$$\Gamma(1/2) = \int_0^\infty e^{-x} . x^{-1/2}\, dx = \int_0^\infty e^{-t^2} . t^{-1} . 2t\, dt = 2\int_0^\infty e^{-t^2} dt = \sqrt{\pi}$$

$t = \sqrt{x}$

$dt = \frac{1}{2\sqrt{x}}\, dx$

$2t\,dt = dx$

ej 6

$\left(-\frac{1}{2}\right)! = \sqrt{\pi}$? :)